CoCoA School 7-12 June 2009
Castelnuovo-Mumford regularity and applications

Maria Evelina Rossi

Università di Genova
Dipartimento di Matematica

TUTORS COCOA: Anna Bigatti and Eduardo De Cabezon Irigaray

Contents

Alternative definitions

One of the aspects that makes the regularity very interesting is that $reg(M)$ can be computed in different ways.

- We say that M is m-regular if

$$reg(M) \leq m$$

($reg(M) := \min\{m : \ M\text{-regular }\}$)

Hence

$$M \text{ is } m\text{-regular} \iff \beta_{ij}(M) = 0 \quad \forall j \geq i + m + 1$$

(equivalently $Tor_i^P(M, k)_j = 0 \quad \forall j \geq i + m + 1$).

In terms of Local Cohomology

The local cohomology module $H^i_m(M)$ with support in m and $0 \leq i \leq d$ ($H^i_m(M) = 0$ $i > d$) is an Artinian graded modules. Let

$$end(H^i_m(M)) := \max\{t : \; H^i_m(M)_t \neq 0\}$$

($\max 0 = -\infty$)

- If $d = dimM$, then

$$reg(M) := \max\{end(H^i_m(M)) + i \;:\; 0 \leq i \leq d\}$$

- By Grothendieck-Serre's formula (Bruns-Herzog Theor. 4.4.3)

$$HP_M(i) - HF_M(i) = \sum_{j=0}^{d} (-1)^{j+1} \lambda(H^j_m(M)_i)$$

  As a consequence

$$HP_M(i) = HF_M(i) \quad \forall i > reg(M)$$

$$reg\text{-}index(M) \leq reg(M)$$

In terms of Ext's

By using the local duality (Eisenbud, A 4.2)

$$H^i_m(M)_t \simeq Ext^{n-i}_P(M,P)_{-t-n}$$

($Ext^j_P(M,P) = H_j(Hom(\mathbb{F},P))$ where $\mathbb{F}$ is a P-free resolution of M)

- (see Eisenbud's book)

$$reg(M) := \min\{m : \; Ext^i_P(M,P)_j = 0 \; : \forall j \leq -m-i-1\}$$

- In the case of k-standard graded algebras P/I

$$reg(M) := \min\{m : \; Ext^i_P(P/I,P)_{-m-i-1} = 0 \}$$

(weakly regularity=regularity)

Regularity and exact sequences

Proposition

Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of graded finitely generated P-modules (homogeneous homomorphisms), then

1) $reg(A) \leq \max(reg(B), reg(C) + 1)$
2) $reg(B) \leq \max(reg(A), reg(C))$
3) $reg(C) \leq \max(reg(A) - 1, reg(B))$
4) *If A has finite length, then* $reg(B) = \max(reg(A), reg(C))$.

Hint: consider the long exact sequence

$$\cdots \to Ext^{j-1}(A,P) \to Ext^{j}(C,P) \to Ext^{j}(B,P) \to$$

$$\to Ext^{j}(A,P) \to Ext^{j+1}(C,P) \to \dots$$

Regularity and linear resolutions

Definition

An ideal I has a pure resolution if for all i the minimal i-syzygies (if any) have all the same degree, that is for all i there exits at most one j so that $\beta_{ij}(I) \neq 0$.

Definition

I has a linear resolution if it is generated in one degree, say d, and $\beta_{ij}(I) = 0$ for all $j \neq i + d$. If this is the case we say that I has d-linear resolution and

$$reg(I) = d.$$

$$0 \to P^{\beta_h}(-d-h) \to \cdots \to P^{\beta_1}(-d-1) \to P^{\beta_0}(-d) \to I \to 0$$

The matrices associated to the maps of the resolution have linear entries.

If I, J have the same HF and both have pure resolution then they have the same Betti numbers.

Regularity and linear resolution

Proposition

Set $I_{\geq k} := I \cap m^k$.

$$r = reg(I) \implies I_{\geq k} \text{ has linear resolution } \forall k \geq r$$

Important fact: If a graded module M has d-linear resolution, then mM has $(d+1)$-linear resolution.

It is enough to consider the exact sequence of graded modules

$$0 \to mM \to M \to M/mM \to 0$$

Now since M is generated in degree d we have $M/mM \simeq k^{\mu}(-d)$ which has d-linear resolution ($reg(K) = 0$). Then by the exact sequence

$$reg(mM) \leq \max\{d, d+1\}$$

On the other hand $reg(mM) \geq d + 1 = indeg\ (mM)$.

```
Use P ::= Q[x,y,z];
I := Ideal(x^2,xy,xz,y^3);
CastelnuovoRegularity(I);
3
-------------------------------
Res(I);
0 --> P(-4) --> P^3(-3)(+)P(-4) --> P^3(-2)(+)P(-3)
-------------------------------

J:=Intersection(I,Ideal(x,y,z)^3);

Res(J);
-------------------------------
0 --> P^3(-5) --> P^9(-4) --> P^7(-3)
```

Regularity and hyperplane sections

Let $F \in P$ be homogeneous such that $0 :_M F$ has finite length, by using the comparison between regularities in exact sequences, we get

$$reg(M) = \max(reg(0 :_M F), reg(M/FM) - \deg F + 1)$$

(Actually it is enough $\dim 0 :_M F \leq 1$)

- If $L \in P_1$ is M-regular, then

$$reg(M) = reg(M/LM)$$

- If L is a linear *filter regular element* ($M_n \xrightarrow{\cdot L} M_{n+1}$ injective $n \gg 0$)

$$reg(M) = \max\{reg(0 : L), reg(M/LM)\} \geq reg(M/LM)$$

(L generic linear form)

Tutorial

Exercise Let $L \in P_1$ be a linear form. Can you find examples with $\mathrm{reg}(I + (L)) > \mathrm{reg}(I)$?

Problem (Caviglia) Is $\mathrm{reg}(I + (L))$ bounded by a polynomial function (possibly quadratic) of $\mathrm{reg}(I)$?

Regularity of a CM module

Proposition

Let M be a Cohen-Macaulay graded finitely generated P-modules of dimension d

1) $reg(M) = \deg(h_M(z))$ *where* $h_M(z)$ *is the* h*-polynomial of* M $(HS_M(z) = \frac{h_M(z)}{(1-z)^d})$
2) $reg(M) = reg\text{-}index(M) + d$

Proof: $(|k| = \infty)$ Let $J = (L_1, \ldots, L_d) \subseteq P$ the ideal generated by a maximal regular sequence of linear forms. We know that

$$reg(M) = reg(M/JM)$$

Now M/JM is an Artinian module and

$$reg(M/JM) = \max\{n : \ (M/JM)_n \neq 0\} = \deg(h_{M/JM}(z)) = \deg(h_M(z))$$

since $HS_M(z) = \frac{h_{M/JM}(z)}{(1-z)^d}$. Hence
$reg(M) = reg\text{-}index(M/JM) = reg\text{-}index(M) + d.$

Regularity and sums, product, intersection of ideals

Let I, J homogeneous ideals, there are the following exact sequences:

$$0 \to P/I \cap J \to P/I \oplus P/J \to P/I + J \to 0$$

$$0 \to I \cap J/IJ \to P/IJ \to P/I \cap J \to 0$$

We prove

Theorem

If $(I \cap J)/IJ$ is a module of dimension at most 1, then

1) $reg(I + J) \leq reg(I) + reg(J) - 1$
2) $reg(I \cap J) \leq reg(I) + reg(J)$
3) $reg(IJ) \leq reg(I) + reg(J)$.

Regularity and sums, product, intersection of ideals

- G. Caviglia gave an example with $\dim(I \cap J)/IJ = 2$ and $reg(I + J) \geq reg(I) + reg(J)$
- The possibility of extending 2) and 3) to any number of ideals is still unclear.
- Conca and Herzog: If $I_1, \ldots, I_r$ are generated by linear forms, then

$$reg(I_1 \cdots I_r) = \sum_i reg(I_i) = r$$

- Derksen and Sidman: If $I_1, \ldots, I_r$ are generated by linear forms, then

$$reg(I_1 \cap \cdots \cap I_r) = \sum_i reg(I_i) = r$$

- Chardin, Cong, Trung: If $I_1, \ldots, I_r$ are monomial complete intersection ideals , then

$$reg(I_1 \cap \cdots \cap I_r) \leq \sum reg(I_i)$$

Tutorial

Exercise Compare the regularities of the ideals I and J with those of $\sqrt{I}$, I^2, $I+J$, IJ in some examples.

Problem Compare $\mathrm{reg}(\sqrt{I})$ and $\mathrm{reg}(I)$ with I is a monomial ideal. Have you some guess in $P = K[x, y]$? In general?
(The answer is known)

Tutorial

Exercise [Chardin-D'Cruz] Let R be the homogeneous coordinate ring of the monomial surface $S \subseteq \mathbb{P}^5$ parametrized by

$$(a^{13}, ab^{12}, a^5c^8, a^5bc^7, a^7b^5c, b^9c^4)$$

Let $I = I(S) \subseteq k[X_0, \ldots, X_5]$ and J the ideal generated by the polynomials of I of degree ≤ 21. By using CoCoA verify that

1. $J = I \cap (X_1, \ldots, X_5)$
2. $\mathrm{reg}(I) = 32$ and $\mathrm{reg}(J) = 24$
3. $\mathrm{depth} P/I = 1$ and $\mathrm{depth} P/J = 2$.

Tutorial

Exercise Let n, m be positive integers and let

$$I_{m,n} = (x^m t - y^m z, z^{n+2} - xt^{n+1}) \subseteq K[x, y, z, t]$$

By CocoA's help, check (in particular cases) the following equalities

1. $\operatorname{reg}(I_{m,n}) = m + n + 2$ (complete intersection)
2. $\operatorname{reg}(\sqrt{I_{m,n}}) = m \cdot n + 2$