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Castelnuovo-Mumford regularity and applications

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Castelnuovo Mumford regularity

The Castelnuovo Mumford regularity

- is one of the most important invariants of a graded module, after the multiplicity and the dimension.
- is related to the theory of syzygies which connects the qualitative study of algebraic varieties and commutative rings with the study of their defining equations.
- is a good measure of the complexity of computing Gröbner bases.
- is a very active area of research which involves specialists working in commutative algebra, algebraic geometry and computational algebra.

Contents

Notations

- Denote

$$P = k[x_1, \dots, x_n]$$

a polynomial ring over a field k with deg $x_i = 1$

$P_j :=$ k-vector space generated by the forms of P of degree j.

- M a finitely generated graded P-module (such as an homogeneous ideal I or P/I), i.e.

$$M = \oplus_i M_i$$

as abelian groups and $P_j M_i \subseteq M_{i+j}$ for every i, j.

Let $d \in \mathbb{Z}$, the d-th twist of M

$$M(d)_i := M_{i+d}.$$

Hilbert Function

Definition

The numerical function

$$HF_M(j) := \dim_k M_j$$

is called the Hilbert function of M.

Assume $M = P/I$ where I is an homogeneous ideal of P.

An important motivation arises in projective geometry.

$X \subseteq \mathbb{P}^r$ a projective variety defined by $I = I(X) \subseteq P = k[x_0, \ldots, x_r]$.

If we write $A(X) = P/I(X)$ for the homogeneous coordinate ring of X :

$$HF_X(d) = \dim_k A(X)_d = \dim_k P_d - \dim_k I_d = \binom{r+d}{r} - \dim_k I_d$$

$\dim_k I_d \dashrightarrow$ the 'number' of hypersurfaces of degree d vanishing on X.

Hilbert Function

If τ is a term ordering on $\mathbb{T}^n$ and $G = \{f_1, \ldots, f_s\}$ is a τ-Gröbner basis of I, then

$$\mathrm{Lt}_\tau\{I\} = \{\mathrm{Lt}_\tau(f_1), \ldots, \mathrm{Lt}_\tau(f_s)\}$$

The residue classes of the elements of $\mathbb{T}^n \setminus \mathrm{Lt}_\tau\{I\}$ form a k-basis of P/I.

Let $\mathrm{Lt}_\tau(I) = (\mathrm{Lt}_\tau(f_1), \ldots, \mathrm{Lt}_\tau(f_s))$.

Proposition

(Macaulay) For every $j \geq 0$

$$HF_{P/I}(j) = HF_{P/\mathrm{Lt}_\tau(I)}(j)$$

Hilbert Polynomial

- $HF_M(j)$ agrees with $HP_M(X)$ a polynomial of degree $d-1$ where $d =$ Krull dimension of M.
- $HP_M(j)$ is called Hilbert Polynomial and it encodes several asymptotic information on M (denote by $e_i(M)$ the Hilbert coefficients).
- A more compact information can be encoded by the Hilbert series

$$HS_M(z) := \sum_{i\geq 0} HF_M(i)z^i = \frac{h_M(z)}{(1-z)^d} \quad \text{(Hilbert – Serre)}$$

where $h_M(1) = e > 0$ is the multiplicity of M and $d = \dim M$.

- Define

$$\text{reg-index}(M) := \max\{i : \quad HF_M(i) \neq HP_M(i)\}$$

Minimal free resolutions

- A graded free resolution of M as a graded P-module is an exact complex ($\ker f_{j-1} = \operatorname{Im} f_j$ for every j)

$$\mathbb{F}: \quad \dots F_h \xrightarrow{f_h} F_{h-1} \xrightarrow{f_{h-1}} \dots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

  where F_i are free P-modules and f_i are homogeneous homomorphisms (of degree 0).

- $\mathbb{F}$ is minimal if for every $i \geq 1$

$$\operatorname{Im} f_i \subseteq mF_{i-1}$$

  where $m = (x_1, \dots, x_n)$.

Existence of minimal graded free resolutions

We proceed step by step:

- Let M be a finitely generated graded P-module. Consider $\{m_1, \ldots, m_t\}$ a minimal system of homogeneous generators of M and let $a_{0i} = \deg m_i$.
- Define the homogeneous map

$$F_0 = \oplus_i P(-a_{0i}) \xrightarrow{f_0} M$$

$$e_i \quad \rightarrow \quad m_i$$

- f_0 is a surjective map and by the minimality of the system of generators

$$\text{Ker } f_0 \subseteq mF_0$$

- Taking a minimal set of generators $\{s_1, \ldots s_r\}$ of Ker f_0 (say of degrees a_{1i}), we define f_1 sending a basis $e'_i \rightarrow s_i$.

$$0 \rightarrow \text{Ker } f_1 \rightarrow F_1 = \oplus_i P(-a_{1i}) \xrightarrow{f_1} Kerf_0 \rightarrow 0$$

we can iterate the procedure.

Minimal free resolution

The minimal graded free resolution of M as P-module has the following shape:

$$\mathbb{F}: \quad \cdots \oplus_{j=1}^{\beta_h} P(-a_{hj}) \xrightarrow{f_h} \oplus_{j=1}^{\beta_{h-1}} P(-a_{h-1j}) \xrightarrow{f_{h-1}} \dots \xrightarrow{f_1} \oplus_{j=1}^{\beta_0} P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

with the properties:

- $a_{ij} \geq i$ for every i, j
- $\forall k \geq 1,\ \forall j = 1, \dots, \beta_k$ there exists p:

$$a_{kj} > a_{k-1p}$$

NO: $\dots P^2(-4) \oplus P(-2) \to P(-3) \oplus P(-2) \to \dots$

- All the non zero entries of the matrices associated to f_i have positive degree

Example

$I = (x^2, xy, xz, y^3)$ in $P = k[x, y, z]$. Define

$$P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0$$

$$e_1 \rightsquigarrow x^2$$
$$e_2 \rightsquigarrow xy$$
$$e_3 \rightsquigarrow xz$$
$$e_4 \rightsquigarrow y^3$$

$Syz_1(I) = \operatorname{Ker} f_0$ is generated by $s_1 = ye_1 - xe_2$; $s_2 = ze_1 - xe_3$; $s_3 = ze_2 - ye_3$; $s_4 = y^2 e_2 - xe_4$. Define

$$P(-3)^3 \oplus P(-4) \xrightarrow{f_1} Syz_1(I) \to 0$$

$$e_i' \rightsquigarrow s_i$$

$Syz_2(I) = \operatorname{Ker} f_1$ is generated by $s = ze_1' - ye_2' + xe_3'$.

A minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \xrightarrow{f_2} P(-3)^3 \oplus P(-4) \xrightarrow{f_1} P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

$$1 \rightsquigarrow s$$

Basic facts I

It will be useful rewrite the resolution as follows:

$$\cdots \to F_i = \oplus_{j\geq 0} P(-j)^{\beta_{ij}} \to \cdots \to \oplus_{j\geq 0} P(-j)^{\beta_{0j}} \to M$$

1) $\beta_{ij} \geq 0$
2) $\beta_{ij} =$ cardinality of the shift $(-j)$ in position i

Question. Does β_{ij} (hence a_{ij}) depend on the maps f_i of the resolution?

We remind that in the proof of the existence of a minimal free resolution we can choose different system of generators of the kernels, hence different maps.

Basic facts I

We prove

Proposition

$$\beta_{ij} = \beta_{ij}(M) = dim_k Tor_i^P(M,k)_j$$

and we call these integers graded Betti numbers of M.

In fact

$$Tor_i^P(M,k) = H_i(\mathbb{F} \otimes P/m)$$

By the minimality of $\mathbb{F}$ the maps of the new complex $\mathbb{F} \otimes P/m$ are trivial, hence we have

$$Tor_i^P(M,k)_j = [\oplus_{m\geq 0} P(-m)^{\beta_{im}} \otimes P/m]_j = [\oplus_{m\geq 0} k(-m)^{\beta_{im}}]_j =$$

$$= \oplus_{m\geq 0}(k_{j-m})^{\beta_{im}} \underset{j=m}{=} k^{\beta_{ij}}$$

Notice that two ideals can have the same HF, but different Betti numbers. $I = (x^2, y^2)$ and $J = (x^2, xy, y^3)$ have both $HF_{P/I} = HF_{P/J} = \{(1,2,1,0)\}$ and different number of generators.

Tutorial

In the tutorial we will see what happens if we consider

$$X = \{P_1, \ldots, P_4\} \subseteq \mathbb{P}^2$$

four distinct points in the plane.

We let $P = k[x_0, x_1, x_2]$:

- the Hilbert polynomial of a set of four points, no matter what the configuration, is a constant polynomial $HP_X(n) = 4$.
- the Hilbert function of X depends only on whether all four points lie on a line.
- The graded Betti numbers of the minimal resolution, in contrast, capture all the remaining geometry: they tell us whether any three of the points are collinear as well.

Basic facts II

We have proved that:

- The graded Betti numbers are uniquely determined by M.
- The minimal graded free resolution is uniquely determined by M up to homogeneous isomorphisms of graded free modules (bases changes).
- The total Betti numbers :

$$\beta_i(M) := \sum_{j \geq 0} \beta_{ij}(M) = rk(F_i)$$

- $\beta_0(M) =$ minimal number of generators of M $(= \dim_k M/mM)$
  $\beta_{0j}(M) = \dim_k M_j/P_1 M_{j-1}$
- $\beta_i(M) =$ number of minimal i-syzygies of M $(= ker\ f_{i-1})$
  $\beta_{ij}(M) =$ number of minimal i-syzygies of M of degree j

Koszul complex

A special graded P-free resolution:

Example

$P = k[x_1, x_2]$ A graded minimal free resolution of $k = P/m$ as P-module is:

$$0 \to P(-2) \to P(-1) \oplus P(-1) \to P \to k \to 0$$

$$1 \to \overline{1}$$

$$e_1 = (1,0) \rightsquigarrow x_1$$

$$e_2 = (0,1) \rightsquigarrow x_2$$

$$1 \rightsquigarrow (-x_2, x_1)$$

More in general we can find a free resolution of $k = P/m$ as $P = k[x_1, \dots, x_n]$-module, $n \geq 1$:

$$\mathbb{K} : 0 \to P(-n)^{\binom{n}{n}} \to P(-n+1)^{\binom{n}{n-1}} \to \cdots \to P(-1)^{\binom{n}{1}} \to P$$

the Koszul complex of $(x_1, \dots, x_n)$.

Hilbert's Syzygy Theorem

We deduce an easy proof of a graded version of

Theorem (Hilbert's Syzygy Theorem)

Every finitely generated P-module has a **finite** *graded free resolution (of length $\leq n$)*

In fact

$$Tor_i(k, M) = H_i(\mathbb{K} \otimes M) = 0$$

for every $i \geq n+1$ ($K_i = 0$ for $i \geq n+1$).

Every graded free resolution $\mathbb{F}$ of M can be minimalized: any free resolution of M can be obtained from a minimal one by adding "trivial complexes" of the form:

$$0 \to \cdots \to P(-a) \to P(-a) \to \cdots \to 0$$

Auslander-Buchsbaum formula

If M has the following minimal P-free resolution:

$$0 \to F_h = \oplus_{j \geq 0} P(-j)^{\beta_{hj}} \to \cdots \to \oplus_{j \geq 0} P(-j)^{\beta_{0j}} \to M$$

Define Projective dimension (or Homological dimension)

$$pd(M) := \max\{i : \beta_{ij}(M) \neq 0 \text{ for some } j\}$$

that is $h =$ length of the resolution.

Theorem (Auslander-Buchsbaum formula)

$$pd_P(M) = n - \mathrm{depth}(M)$$

where $\mathrm{depth}(M) =$ *length of a (indeed any) maximal M-regular sequence in m.*

M is Cohen-Macaulay $\iff$ $\mathrm{depth} M = \dim M$ $\iff$ $\mathrm{pd}_P(M) = n - \dim M$.

Let I be an homogeneous ideal of P.

Proposition

The Betti numbers of I determine the HF of I. If β_{ij} are the graded Betti numbers of I, then the Hilbert series of P/I is given by

$$HS_{P/I}(z) = \frac{1 + \sum_{ij}(-1)^{i+1}\beta_{ij}z^j}{(1-z)^n}$$

If we consider the previous example $I = (x^2, xy, xz, y^3)$ in $P = k[x, y, z]$. We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \to P(-3)^3 \oplus P(-4) \to P(-2)^3 \oplus P(-3) \to P \to P/I \to 0.$$

Since $HS_{P(-d)^\beta}(z) = \frac{\beta z^d}{(1-z)^n}$, then

$$HS_{P/I}(z) = \frac{1 - 3z^2 - z^3 + 3z^3 + z^4 - z^4}{(1-z)^3} = \frac{1+2z}{1-z}$$

Betti Diagram

The numerical invariants in a minimal free resolution can presented by using "a piece of notation" introduced by Bayer and Stillman: the Betti diagram.

This is a table displaying the numbers β_{ij} in the pattern

	0	1	2	...	i
0 :	β_{00}	β_{11}	β_{22}	$\cdots$	β_{ii}
1 :	β_{01}	β_{12}	β_{23}	$\cdots$	β_{ii+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s :	β_{0s}	β_{1s+1}	β_{2s+2}	$\cdots$	β_{ii+s}
$\sum$	β_0	β_1	β_2	$\cdots$	β_i

with β_{ij} in the i-th column and $(j-i)$-th row.

Thus the i-th column corresponds to the i-th free module

$$F_i = \oplus_j P(-j)^{\beta_{ij}}.$$

Example

```
  Use R ::= QQ[t,x,y,z];
  I := Ideal(x^2-yt,xy-zt,xy);
  Res(I);
0 --> R^2(-5) --> R^4(-4) --> R^3(-2)
-------------------------------
  BettiDiagram(I);
         0    1    2
 --------------------
  2:     3    -    -
  3:     -    4    2
 --------------------
 Tot:    3    4    2
 -------------------------------
```

Definition

Given a minimal P-free resolution of M :

$$\mathbb{F} : \ldots\ldots \quad \to F_i = \oplus P(-j)^{\beta_{ij}(M)} \to \cdots \to F_0 = \oplus P(-j)^{\beta_{0j}(M)}$$

the Castelnuovo-Mumford regularity of M is

$$reg(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$$

We remark that if I is an homogeneous ideal $\subseteq P$

$$pd(P/I) = pd(I) + 1$$

$$reg(I) = reg(P/I) + 1$$

Moreover:

- $reg(I) \geq$ maximum degree of a (minimal) generator
- if M is Artinian

$$reg(M) = \max\{i : M_i \neq 0\}$$

Exercise. Starting from the Betti Diagram, write a CocoA function returning the Castelnuovo regularity of M.

```
 Use P ::= Q[x,y,z,w];
 I := Ideal(xz-yw, xw-y^2,  x^2y+xzw, xy^2, xyz);
 CastelnuovoRegularity(I);
 4
 -------------------------------
 Res(I);
-------------------------------
P^2(-7) -> P^6(-6) -> P^5(-4)(+)P^3(-5)-> P^2(-2)(+)P^3(-3)
-------------------------------
BettiDiagram(I);
-------------------------------
        0    1    2    3
-------------------------
 2:     2    -    -    -
 3:     3    5    -    -
 4:     -    3    6    2
-------------------------
Tot:    5    8    6    2
-------------------------------
```

If we consider THE example

$$I = (x^2, xy, xz, y^3) \subseteq P = k[x, y, z].$$

We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to F_2 = P(-4) \xrightarrow{f_2} F_1 = P(-3)^3 \oplus P(-4) \xrightarrow{f_1} F_0 = P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

Then

- $pd(I) = 2$
- $reg(I) = 3 =$ max degree of a minimal generator.
- $\dim P/I = 1$ (we know that $HS_{P/I}(z) = \frac{1+2z}{1-z}$).

  Hence P/I is not Cohen-Macaulay since $pd(P/I) = 3 > 3 - \dim P/I = 2$.
- $reg\text{-}index(P/I) < reg(P/I) = 2$

Lex-segment ideal

Let I be an homogeneous ideal in $P = k[x_1, \ldots, x_n]$.

By Macaulay's Theorem there exists a lexicographic ideal L with the same HF of I (L_j is spanned by the first $\dim_K L_j = \dim_K I_j$ monomials in the lexicographic order).

- (Bigatti, Hulett, Pardue)

$$\beta_{ij}(P/I) \leq \beta_{ij}(P/L)$$

- Hence $reg(P/I) \leq reg(P/L)$
- (I. Peeva) the Betti numbers $\beta_{ij}(P/I)$ can be obtained from $\beta_{ij}(P/L)$ by a sequence of consecutive cancellations.

  i.e. $\cdots \to P(-6)^2 \oplus P(-5) \to P(-5) \oplus P(-3) \to \ldots$

Tutorial

Exercise Consider the homogeneous coordinate ring of the "twisted cubic":

$$R = K[s^3, s^2t, st^2, t^3]$$

1. Prove that $R = P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$
2. Prove that R is CM
3. Compute $\mathrm{HF}_R(j)$, $\mathrm{reg}(R)$
4. Compare $\mathrm{reg}(I)$ and $\mathrm{reg}(Lt_\tau(I))$ with τ any term ordering

Exercise Consider the homogeneous coordinate ring of the smooth rational quartic in $\mathbb{P}^3$

$$R = K[s^4, s^3t, st^3, t^4]$$

1. Prove that $R \simeq P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1^2 & x_1x_3 & x_2 \\ x_1 & x_0x_2 & x_2^2 & x_3 \end{pmatrix}$
2. Prove that R is not CM
3. Compute $\mathrm{reg}(I)$

Tutorial

Exercise Compute the Betti diagram of 11 randomly chosen points in $\mathbb{P}^7$. Compute regularity index (`RegularityIndex`) and regularity.

Exercise Let $P = K[x_1, \ldots, x_n]$ and $F_1, F_2, F_3 \in P$ homogeneous polynomials which form a regular sequence.

1. Assume $d_i = \deg(F_i)$ and compute $\mathrm{reg}(I)$ where $I = (F_1, F_2, F_3)$
2. Can you compute the value of $\mathrm{reg}(I)$ where I is generated by a regular sequence of degrees $d_1, \ldots, d_r$?

Exercise Describe Hilbert function, Hilbert polynomial, Betti diagram, regularity of each possible configuration of 4 distinct points in $\mathbb{P}^2$.