# Hilbert Functions of Points, Thin and Fat

We saw last time that the dimension of the Secant Varieties of the Veronese varieties depends on the knowledge of the Hilbert function of intersections of ideals of the type $\wp^2$, where $\wp$ is the ideal of a point in $\mathbb{P}^n$. More precisely we showed,

**Theorem:** If $P_0, P_1, \ldots, P_s$ are $s+1$ general points of $\mathbb{P}^n$, which correspond to the prime ideals $\wp_0, \ldots, \wp_s$ of $R = k[x_0, \ldots, x_n]$ then

$$\dim Sec_s(\nu_j(\mathbb{P}^n)) = H(R/\wp_0^2 \cap \cdots \cap \wp_s^2, j) - 1$$

What I would like to do in this lecture is discuss the Hilbert function of ideals of the type

$\wp_0 \cap \cdots \cap \wp_s$ and $\wp_0^2 \cap \cdots \cap \wp_s^2$ where the $\wp_i$ are the ideals of points in $\mathbb{P}^n$.

As I've noted earlier, if $P = [a_0 : \ldots : a_n]$ is a point in $\mathbb{P}^n$ then the ideal $I = (L_1, \ldots, L_n)$, generated by $n$ linearly independent linear forms which vanish on $P$, is precisely the ideal in $R$ of all the functions vanishing on $P$.

If, moreover, $P_0, \ldots, P_s$ are $s+1$ points of $\mathbb{P}^n$ and $P_i \leftrightarrow \wp_i$ then if

$$I = \wp_0 \cap \cdots \cap \wp_s$$

we have $Z(I) = \{P_0, \ldots, P_s\}$. If we set $A = R/I$, then

$$H(A, d) = \dim R_d - \dim I_d = \binom{d+n}{n} - \dim_k I_d$$

and so we will know $H(A, d)$ if and only if we know $\dim_k I_d$.

So, let $F \in R_d$, then

$$F = a_1 M_1 + \cdots + a_{\binom{d+n}{n}} M_{\binom{d+n}{n}}$$

where the $M_i$ are all the monomials of degree $d$ in $R$.

If $P_i = [a_{i0} : \cdots : a_{in}]$, then $F(P_i) = 0$ if and only if

$$a_1 M_1(P_i) + \cdots + a_{\binom{d+n}{n}} M_{\binom{d+n}{n}}(P_i) = 0.$$

Now, this last is a linear equation in the unknowns $a_1, \ldots, a_{\binom{d+n}{n}}$, so if we write such an equation for each of the points $P_0, \ldots, P_s$ we get a system of $s+1$ linear equations which we can write as

$$\begin{bmatrix} M_1(P_0) & \cdots & \cdots & M_{\binom{d+n}{n}}(P_0) \\ \cdots & & & \cdots \\ \cdots & & & \cdots \\ M_1(P_s) & \cdots & \cdots & M_{\binom{d+n}{n}}(P_s) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ \\ \vdots \\ a_{\binom{d+n}{n}} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \vdots \\ \\ 0 \end{bmatrix}$$

and $(a_1, \ldots, a_{\binom{d+n}{n}})$ is a solution to this system if and only if

$$F = a_1 M_1 + \cdots + a_{\binom{d+n}{n}} M_{\binom{d+n}{n}}$$

vanishes at all the $P_i$.

Now, the number of independent solutions to this system of equations is

$$\dim_k I_d = \binom{d+n}{n} - \text{ the rank of the coefficient matrix } \mathcal{M},$$

where the coefficient matrix is

$$\mathcal{M} = \begin{bmatrix} M_1(P_0) & \cdots & \cdots & M_{\binom{d+n}{n}}(P_0) \\ \cdots & & & \cdots \\ \cdots & & & \cdots \\ M_1(P_s) & \cdots & \cdots & M_{\binom{d+n}{n}}(P_s) \end{bmatrix}.$$

Rewriting that equation we have

$$rk\ \mathcal{M} = \binom{d+n}{n} - \dim_k(I_d) = H(R/I, d)$$

Thus we have that

$$H(R/I, d) = rk\ \mathcal{M} \leq \min\{s+1, \binom{d+n}{n}\}$$

where $s+1$ is the number of points we are considering.

**Definition:** We say that a set $\mathbb{X}$ of $s+1$ points in $\mathbb{P}^n$ has *generic Hilbert function* if

$$H(R/I_{\mathbb{X}}, d) = \min\{s+1, \binom{d+n}{n}\} \text{ for every } d.$$

> **Note:** It is not hard to show that almost all sets of $s+1$ points of $\mathbb{P}^n$ have generic Hilbert function.

Now, since $A = R/\wp_0 \cap \cdots \cap \wp_s$ we know that

1) $A$ is a 1-dimensional ring;

2) Let $F \in R$ be homogeneous. Then $\overline{F} \in A$ is not a zero divisor if and only if $F \notin \wp_i$ for any $i$ i.e. if the hypersurface defined by $F$ does not contain *any* of the points $P_i$.

In particular, if $k$ is an infinite field then there is a linear form $L$ which is not in $\cup_{i=0}^{s} \wp_i$ and hence $\overline{L}$ is not a zero divisor in $A$.

Such an $L$ defines a linear transformation, given by multiplication by $L$,

$$\overline{L} : A_t \longrightarrow A_{t+1}$$

for every $t$, which is injective (since $L$ is not a zero divisor) and hence

$$\dim_k A_t \le \dim_k A_{t+1}.$$

Moreover, since $A/\overline{L}A$ is a standard graded algebra, if we have equality above, for some $t$, then we have equality for all $\ell \ge t$ (explain).

It is also the case that $\dim_k A_s = s + 1$. To see why that is so, let $L_i$ be a linear form with the property that $L_i(P_j) = \delta_{i,j}$. Such $L_i$ exist for $i = 0, \ldots, s$.

Consider the set of $s + 1$ forms of degree $s$

$$F_0 = L_1 \cdots L_s,\ F_1 = L_0 L_2 \ldots L_s, \cdots, F_s = L_0 L_1 \cdots L_{s-1}$$

It is clear that

$$F_i(P_j) = \begin{cases} 0 & \text{if } i \ne j \\ \ne 0 & \text{if } i = j \end{cases}$$

Thus all these $F_i$ are not in $\wp_0 \cap \cdots \cap \wp_s$. But, more is true!

**Claim:** No linear combination of the $F_i$'s is in $I = \wp_0 \cap \cdots \cap \wp_s$.

*Proof:* Suppose that $\alpha_0 F_0 + \cdots + \alpha_s F_s \in I$. Then

$$\alpha_0 F_0(P_0) + \cdots + \alpha_s F_s(P_0) = 0.$$

But, since $F_j(P_0) = 0$ for $j = 1, \ldots, s$ we get $\alpha_0 F_0(P_0) = 0$. Since $F_0(P_0) \neq 0$ we must have $\alpha_0 = 0$. One continues in this way. $\square$

It follows that $\dim_k A_s \geq s + 1$. Since $\dim_k A_s \leq s + 1$ the result follows.

The last thing I want to mention about the Hilbert function of $A$ is that the Hilbert function of $A/\overline{L}A$ is the first difference of the Hilbert function of $A$ and so that first difference is governed by Macaulay's Theorem. It turns out that these are the only conditions necessary to describe the Hilbert functions of sets of $s + 1$ points in $\mathbb{P}^n$, more precisely:

**Theorem:** Let $\{a_0 = 1, a_1 \leq n+1, a_2, \ldots\}$ be an infinite sequence of non-negative integers. Suppose that

1) these integers satisfy Macaulay's growth condition in order to be an O-sequence;

2) $a_i = s+1$ for all $i >> 0$;

3) the first difference sequence $\{1, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots\}$ also satisfies Macaulay's growth condition.

Then there is a set of $s+1$ points $\mathbb{X}$ of $\mathbb{P}^n$ such that

$$H(R/I_{\mathbb{X}}, t) = a_t.$$

What about the Hilbert functions of ideals of the form $\wp_0^2 \cap \cdots \cap \wp_s^2$?

These ideals are called 2-*fat points* because they are defined by primary ideals (namely $\wp^2$) for the ideals of points. Although we can say something about the Hilbert functions of these ideals, we know much less about them than we do for the Hilbert function of the ideal of simple points.

Let me begin by looking at one $\wp$.

We may as well assume that $\wp = (x_1, \ldots, x_n)$ and that $P = [1 : 0 : \ldots : 0]$, i.e. $F(P) = 0$ if and only if $F \in \wp$.

Now, recall that we can break up $\mathbb{P}^n$ into the disjoint union of $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$, i.e

$$\mathbb{P}^n = \{[a_0 : \ldots : a_n] \mid a_0 \neq 0\} \cup \{[a_0 : \ldots : a_n] \mid a_0 = 0\} = \mathbb{Y}_1 \cup \mathbb{Y}_2$$

where $\mathbb{Y}_1 = \mathbb{A}^n$ and $\mathbb{Y}_2 = \mathbb{P}^{n-1}$.

We have $P \in \mathbb{Y}_1$ if and only if $P = [1 : \frac{a_1}{a_0} : \cdots : \frac{a_n}{a_0}] = (\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0}) = (b_1, \ldots, b_n) \in \mathbb{A}^n$. Clearly $F(P) = 0$ if and only if $F(1, b_1, \ldots, b_n) = 0$.

Consider the not necessarily homogeneous polynomial $F(1, x_1, \ldots, x_n)$. It is called the *dehomogenization of* $F$ *with respect to* $x_0$. Let's write it as

$$f = f_0 + f_1(x_1, \ldots, x_n) + f_2(x_1, \ldots, x_n) + \cdots + f_r(x_1, \ldots, x_n).$$

with $f_i$ homogeneous of degree $i$.

Now $F(1, 0, \dots, 0) = 0$ if and only if $f(0, \dots, 0) = 0$, i.e if and only if $f_0 = 0$. I.e. vanishing at a point imposes one condition on $F$.

Now, $F \in \wp^2$ if and only if $F(P) = 0$, i.e. $f_0 = 0$ **and also** $f_1(x_1, \dots, x_n) = 0$, i.e. the $n$ coefficients of $f_1$ all have to be 0. Thus, for $F \in \wp^2$ we need to impose $n + 1$ conditions on $F$.

But, the coefficients of $f_1$ are (by Taylor's Theorem) nothing more than the partial derivatives of $F$ evaluated at the point $P$. So, we are saying that all the first partial derivatives of $F$ vanish at the point $P$.

We can look at this another way: writing down a form of degree $F$ with unknown coefficients and imposing that it vanish at a point and that all of its first derivatives vanish at the same point is exactly $n + 1$ linearly independent conditions on the form $F$.

What I am saying is that

$$H(R/\wp^2, -) := 1 \quad n+1 \quad n+1 \quad \rightarrow$$

The statement about the vanishing of all the first partial derivatives is another way to say that $F \in \wp^2$ if and only if the hypersurface $Z(F)$ has a singularity at the point $P$.

More generally, if $\wp_i \leftrightarrow P_i \in \mathbb{P}^n$, then $F \in \wp_0^2 \cap \cdots \cap \wp_s^2$ if and only if $Z(F)$ is a hypersurface with a singularity at all the points $P_i$.

So, if we assume that having a singular point at $P_i$ is independent of having one at $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_t$ for every $i$. We expect that the number of conditions in order to have a form of degree $d$ be in $\wp_1^2 \cap \cdots \cap \wp_t^2$ , is

$$\min\{t(n+1), \dim R_d\}$$

i.e. we expect that

$$H(R/(\wp_1^2 \cap \cdots \cap \wp_t^2), d) = \min\{t(n+1), \dim R_d\}.$$

If that is the case, then

$$H(R/(\wp_1^2 \cap \cdots \cap \wp_t^2), d) - 1 = \min\{t(n+1), \dim R_d\} - 1$$

$$= \min\{tn + (t-1), \dim \mathbb{P}^\ell\} \quad \ell = \binom{d+n}{n} - 1\}$$

$$= \text{ the expected dimension of } Sec_t(\nu_d(\mathbb{P}^n)).$$

We have proved the following:

If $R = k[x_0, \dots, x_n]$, the condition that

$$H(R/\wp_0^2 \cap \cdots \cap \wp_s^2, d) = \min\{(s+1)(n+1), \binom{d+n}{n}\}$$

is equivalent to the statement that $Sec_s(\nu_d(\mathbb{P}^n))$ has the expected dimension.

Let's look at a few examples:

**Example 1:** Consider $\nu_4(\mathbb{P}^1) \subset \mathbb{P}^4$. We have

$$\nu_4 : [x_0 : x_1] \longrightarrow [x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4] \in \mathbb{P}^4$$

We know that

$$\dim(Sec_1(\nu_4(\mathbb{P}^1))) = H(R/(\wp_1^2 \cap \wp_2^2), 4) - 1.$$

Since there are only two points in this case we can suppose that

$$P_1 = [1:0] \leftrightarrow \wp_1 = (x_1),\ P_2 = [0:1] \leftrightarrow \wp_2 = (x_0).$$

So,

$$\wp_1^2 \cap \wp_2^2 = (x_1^2) \cap (x_0^2) = (x_1^2 x_0^2)$$

Thus,

$$\dim \left(k[x_0, x_1]/\wp_1^2 \cap \wp_2^2\right)_4 = 5 - 1 = 4$$

and so

$$\dim(Sec_1(\nu_4(\mathbb{P}^1))) = 4 - 1 = 3.$$

We can use this argument to prove the more general result

**Proposition:** $\dim Sec_1(\nu_j(\mathbb{P}^1)) = 3$ for all $j \geq 4$.

*Proof:* From what we saw above, it's enough to show that

$$\dim_k \left(k[x_0, x_1]/\wp_1^2 \cap \wp_2^2\right)_j = 4$$

This is easy to show since, wlog we can assume $\wp_1 = (x_0)$ and $\wp_2 = (x_1)$ and that

$$\dim_k \left(k[x_0, x_1]/\wp_1^2 \cap \wp_2^2\right)_4 = 4.$$

Notice that $\overline{L} = \overline{x_0 + x_1}$ is not a zero divisor in the ring

$$A = k[x_0, x_1]/\wp_1^2 \cap \wp_2^2,$$

so

$$\dim_k \left(k[x_0, x_1]/\wp_1^2 \cap \wp_2^2\right)_j \geq 4$$

for every $j$ and so the secant variety has dimension $\geq 3$. But, the only problem was if the dimension were less than 3!

Another way to see this is to notice that $(x_0^2x_1^2)$ is a monomial ideal and a basis for the quotient of this ideal, in degree $j \geq 4$, is given by the four monomials

$$x_0^j, x_0^{j-1}x_1, x_0x_1^{j-1}, x_1^j.$$

That is enough to finish the proof.

**Remark:** It turns out that we can prove more. Namely

**Theorem:**

$$Sec_t(\nu_j(\mathbb{P}^1))$$

always has the expected dimension.

*Proof:* Let $P_0, \ldots, P_t$ be $t+1$ general points on the projective line and let $P_i \leftrightarrow \wp_i$ a prime ideal in $k[x_0, x_1]$. It is easy to see that $\wp_i = (L_i)$ and hence that $\wp_i^2 = (L_i^2)$.

Thus $\wp_0^2 \cap \cdots \cap \wp_t^2 = (L_0 L_1 \cdots L_t)^2$, which is a principal ideal. Now,

$$\dim Sec_t(\nu_j(\mathbb{P}^n)) = \dim(R/\wp_0^2 \cdots \cap \wp_t^2)_j - 1$$

But, when $j \geq 2t+1$ we have that

$$\dim R_j = j+1 \text{ and } \dim(L_0 L_1 \cdots Ł_t)_j^2 = j-(2t+2)+1.$$

Thus, the dimension of the quotient is: $(j+1)-(j-(2t+2)+1) = 2t+2$, and $2t+1 = (2t+2)-1$ is exactly the expected dimension of the secant variety.

**Example:** The variety $Sec_4(\nu_4(\mathbb{P}^2))$ is defective.

In this case we have $R = \mathbb{C}[x_0, x_1, x_2]$ and

$$\mathbb{P}^2 = \mathbb{P}(R_1) \xrightarrow{\nu_4} \mathbb{P}(R_4) \simeq \mathbb{P}^{14}.$$

The expected dimension of this secant variety is

$$\min\{5 \cdot 2 + 4, 14\} = 14$$

In order for that to actually be the dimension we must have:
for $P_0, P_1, P_2, P_3, P_4$ (5 general points of $\mathbb{P}^2$) corresponding the the prime ideals $\wp_0, \ldots, \wp_4$ that

$$H(R/\wp_0^2 \cap \cdots \cap \wp_4^2, 4) = 15 = \dim_{\mathbb{C}} R_4$$

i.e.

$$(\wp_0^2 \cap \cdots \cap \wp_4^2)_4 = (0) \ .$$

But, there is always a quadric $Q$ through 5 points of $\mathbb{P}^2$ (explain) and so $Q^2$, a form of degree 4, is singular at the 5 points. Thus

$$\dim Sec_4(\nu_4(\mathbb{P}^2)) < 14.$$

It is easy to show, using exactly the same kind of argument (explain), that

$$Sec_8(\nu_4(\mathbb{P}^3)) \text{ and } Sec_{13}(\nu_4(\mathbb{P}^4))$$

are defective.

It turns out that

$$Sec_6(\nu_3(\mathbb{P}^4)) \subset \mathbb{P}^{34}$$

is also defective.

The incredible fact is the following:

**Theorem:** (J. Alexander - A. Hirschowitz) Let $j \geq 3$ and let

$$\mathbb{X} = Sec_t(\nu_j(\mathbb{P}^n)) \ .$$

Then, apart from the four exceptions mentioned above (and the quadratic Veronese varieties, i.e. for $j = 2$), all these secant varieties are not defective.

Interestingly enough, the proof never discusses secant varieties! It resolves the more general question:

given $P_0, \ldots, P_t$ general points in $\mathbb{P}^n$ with defining prime ideals $\wp_0, \ldots, \wp_t$ in $R = \mathbb{C}[x_0, \ldots, x_n]$. Let $I = \wp_0^2 \cap \cdots \cap \wp_t^2$ and set $A = R/I$.

Then, for $j \geq 3$, we have

$$H(R/I, d) = \min\{t(n+1), \dim_{\mathbb{C}} R_d\}$$

for all $d$ and $t$ except for the four cases mentioned above.

This is one of the most important single results on secant varieties for almost 50 years and the features of it that I want to emphasize are the following:

$a$) It is a *complete* characterization of all the defective secant varieties for the family of Veronese varieties.

$b$) That characterization has the following special features:

$i$) There is an infinite family of deficient varieties for which the deficiency is easy to understand (in this case the quadratic Veronese varieties).

*ii*) There is a short list of exceptions, each of which has a reasonable explanation for the deficiency.

*iii*) The exceptions all occur for "small" numbers.